Why schemes?

The concept of a scheme can seem arbitrary and contrived without any context, so let's review our basic AG dictionary so we can see exactly <u>what</u> it attempts to generalize, and <u>why</u>.

Let k be an algebraically closed field, and $S = k[x_1, \dots, x_n]$.

The geometry of affine varieties is directly tied to the algebra of the ring S and its quotients. Recall the following dictionary between the geometry and the commutative algebra:

AG		comm. alg.
Algebraic sets in $\mathbb{A}^n$ (w/ coord ring S/I)	$\longleftrightarrow$	radical ideals $I \subseteq S$
Affine algebraic varieties in $\mathbb{A}^n$	$\longleftrightarrow$	prime ideals $P \subseteq S$
points in $\mathbb{A}^n$ $(a_1, \dots, a_n)$	$\longleftrightarrow$	maximal ideals in S $(x_1 - a_1, \dots, x_n - a_n)$
$\emptyset$	$\longleftrightarrow$	$(1) = S$

$\mathbb{A}^n$	$\longleftrightarrow$ (0)
Inclusion of alg. sets	$\longleftrightarrow$ (reverse) inclusion of ideals
Irreducible hypersurfaces	$\longleftrightarrow$ Irr. polynomials (up to scaling)
Alg. subsets of $V = V(I)$ (I radical)	$\longleftrightarrow$ Radical ideals in $\Gamma(V) = S/I$ ($\longleftrightarrow$ radical ideals in S containing I)
Points in $V = V(I)$	$\longleftrightarrow$ max'l ideals in S/I
Regular maps $V \to W$ (V, W affine varieties)	$\longleftrightarrow$ k-algebra homomorphisms $\Gamma(W) \to \Gamma(V)$
Varieties in $\mathbb{A}^n$ containing a point P	$\longleftrightarrow$ Prime ideals in $\mathcal{O}_P(\mathbb{A}^n)$

(Homework: Review any of these correspondences that are unfamiliar to you!)

We get varieties more generally by "glueing" affine varieties along open sets, e.g. $\mathbb{P}^n$ and projective varieties.

Key idea: varieties can be reduced to their affine covers, and an affine variety V is completely determined up to isom. by its coord. ring $\Gamma(V)$, some finitely-generated k-algebra, integral domain. (In fact, they are categorically anti-equivalent.)

This choice of ring is natural, because it allows us to describe affine varieties as the vanishing locus of polys. However, we can generalize this to affine schemes by replacing the k-algebra w/ an arbitrary commutative ring R, and instead of looking at vanishing loci, defining everything in terms of ideals.

Note that R is allowed to have zero-divisors and nilpotents, which allows for nonreduced structure.

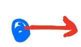
Ex: Consider $I = (x^2, y) \subseteq k[x,y]$. Using our usual definitions, $V(I) = V(\sqrt{I}) = V(x,y) = (0,0) \in \mathbb{A}^2$.

However, we will see that I and $\sqrt{I}$ define two different "schemes":

scheme corr. to (x,y):

$(0,0)$

scheme corr. to (x^2,y):



$(0,0)$ + tangent direction

This generalization also allows us to associate seemingly purely algebraic rings, such as $\mathbb{Z}[\sqrt{2}]$, with a geometric structure.